THE UNIVERSITY of York

Discussion Papers in Economics

No. 12/02

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Abstract: We consider a market model in which all commodities are inherently indivisible and thus are traded in integer quantities. We ask whether a finite set of price-quantity observations satisfying the Generalized Axiom of Revealed Preference (GARP) is consistent with utility maximization. Although familiar conditions such as non-satiation become meaningless in the current discrete model, by refining the standard notion of demand set we show that Afriat's celebrated theorem still holds true. Exploring network structure and a new and easy-to-use variant of GARP, we propose an elementary, simple, intuitive, combinatorial, and constructive proof for the result.

Keywords: Afriat's theorem, GARP, indivisibilities, revealed preference.

JEL classification: D11, C60.

1 Introduction

The theory of demand typically assumes that all commodities in the market are perfectly divisible, and a consumer, when faced with prices and a budget, will choose an affordable bundle to achieve a maximal utility. In a pioneering article, Afriat (1967) started with a finite set of observed market prices and the consumer's demand quantities and asked whether such observations are actually consistent with the maximization of a locally non-satiated utility function. By induction he established a remarkable result stating that the observations are consistent with utility maximization if and only if they satisfy the Generalized Axiom of Revealed Preference—a simple testable condition. This work has stimulated considerable interest and substantial follow-up research; see Diewert (1973), Varian (1982), Blundell, Browning and Crawford (2003), Fostel, Scarf and Todd (2004), Piaw and Vohra (2003), Cherchye, De Rock and Vermeulen (2007) among many others. See Varian (2006) and Vermeulen (2011) for more references.

[1] January 12, 2012 (This version). Part of this research was done while the second author was visiting the Research Institute for Mathematical Sciences, Kyoto University, Japan. The author wishes to thank the institute for its hospitality and financial support.

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While the literature focuses on the case of divisible goods, the current paper attempts to extend the theory to an equally important and practical case in which all commodities are traded in integer quantities, for instance, when all goods are inherently indivisible. In reality, indivisible commodities are pervasive and constitute significant parts of many economies. In general, they are durable and expensive, to name but a few, such as houses, cars, computers, machines, arts, employees, and airplanes. In fact, many divisible goods are also traded in discrete quantities, such as oil sold in barrels. Obviously, modeling economies with indivisibilities is more meaningful and more realistic. The importance of studying such economies has long been recognized by many economists, including Lerner (1944), Koopmans and Beckmann (1957), Debreu (1959), Arrow and Hahn (1971), Shapley and Scarf (1974), Kelso and Crawford (1982), and Scarf (1986, 1994). In the current environment, due to absence of perfect divisibility and continuity, familiar conditions such as non-satiation can no longer be applied. To tackle the problem, we need to refine the standard concept of demand set. Using this refinement, we will be able to show that Afriat's theorem still holds true in the current discrete case. This demonstrates surprisingly wide appeal and adaptability of Afriat's theorem. We also introduce an easy-to-use variant of the Generalized Axiom of Revealed Preference. Using network structure and the new variant of GARP, we present a very elementary, simple, intuitive, combinatorial and constructive proof for the result. The basic idea of the proof was used explicitly in Piaw and Vohra (2003) and also implicitly in Afriat (1967), Diewert (1973), Varian (1982), and Fostel, Scarf and Todd (2004). Here we improve the argument considerably and make it very transparent and accessible without assuming the reader's familiarity with any fundamental result from graph theory, linear programming, or any other mathematical subject. The proof is so easy that it can be understood by college economics students. In addition the proof is not restricted to indivisible goods and can be equally applied to divisible goods.

2 Main Results

We begin by reviewing the purchase decision problem of a consumer. There are n different types of commodities in the market. The consumer has a budget b for consumption and a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$.[4] Suppose $p \in \mathbb{R}^n_+$ are the prevailing market prices, each component p_i indicating the price of commodity i. Then the consumer's decision problem is to choose a bundle x in $\mathbb{R}^n_+$ which gives him the highest utility and is also affordable to him. Such a bundle is called an *optimal bundle*. Alternatively, we can describe all his optimal bundles by using the demand set $D_u(p, b) = \arg\max\{u(x) \mid p \cdot x \leq b,\ x \in \mathbb{R}^n_+\}$.

[4]Here $\mathbb{R}^n_+$ denotes the nonnegative orthant of the n-dimensional Euclidean space $\mathbb{R}^n$. We use $\mathbb{Z}^n$ and $\mathbb{Z}^n_+$ to stand for the set of all integral vectors in $\mathbb{R}^n$ and $\mathbb{R}^n_+$, respectively.

In the literature it is typically assumed that all commodities are perfectly divisible and also the consumer's utility function u is *locally non-satiated* in the sense that for every $x \in \mathbb{R}^n_+$, and in every neighborhood of x, there is another bundle having a higher utility. Suppose that a market analyst wishes to examine the consumer's demand behavior. It is natural to assume that the analyst does not know the consumer's utility function and his budget flow but does know that the consumer does not change his preferences over a period of time. Suppose that the analyst has now collected a finite observed data set $\{(p^i, x^i) \mid i = 1, \cdots, m\}$ with respect to the consumer over the time $i = 1, \cdots, m$, where $p^i \in \mathbb{R}^n_+$ is the price vector and $x^i \in \mathbb{R}^n_+$ is the consumer's demand bundle under prices p^i and (probably an unobservable) budget b_i (which may vary over the time). The fundamental question raised by Afriat (1967) is whether these observations are consistent with the consumer's demand behavior under a locally non-satiated utility function u in the sense that $x^i \in D_u(p^i, b_i)$ for all $i = 1, \cdots, m$. To verify the consistency, several criteria have been proposed. Among them, the Strong Axiom of Revealed Preference (SARP) and the Generalized Axiom of Revealed Preference (GARP) are most well-known and widely used.

A consumer's choice behavior is said to satisfy the Strong Axiom of Revealed Preference (SARP) if, for every sequence of pairs of price vector and demand bundle (p^1, x^1), (p^2, x^2), $\cdots$, (p^m, x^m) satisfying $p^j \cdot x^{j+1} \leq p^j \cdot x^j$ for all $j \leq m-1$, we have $p^m \cdot x^1 > p^m \cdot x^m$. SARP was due to Houthakker (1950).[5] Moreover, we say that the consumer's behavior satisfies the Generalized Axiom of Revealed Preference (GARP) if, for every sequence of pairs of price vector and demand bundle (p^1, x^1), $\cdots$, (p^m, x^m) satisfying $p^j \cdot x^{j+1} \leq p^j \cdot x^j$ for all $j \leq m-1$, we have $p^m \cdot x^1 \geq p^m \cdot x^m$. GARP is more general than SARP and was introduced in Varian (1982).[6]

Given a finite observed data set $\{(p^i, x^i) \mid i \in M\}$, where $M = \{1, 2, \cdots, m\}$, $p^i \in \mathbb{R}^n_+$ is a price vector and $x^i \in \mathbb{R}^n_+$ is the corresponding demand bundle, we say that a utility function u **rationalizes** the observed behavior if the data can be generated as the outcome of the utility-maximization, i.e., $x^i \in D_u(p^i, b_i)$ for some b_i and for all i. We also say that the data set $\{(p^i, x^i) \mid i \in M\}$ satisfies GARP if, for every subset $\{(p^{i_j}, x^{i_j}) \mid j = 1, \cdots, t\}$ of the data set $\{(p^i, x^i) \mid i \in M\}$, $p^{i_j} \cdot x^{i_{j+1}} \leq p^{i_j} \cdot x^{i_j}$ for all $j \leq t-1$ implies $p^{i_t} \cdot x^{i_1} \geq p^{i_t} \cdot x^{i_t}$. Afriat (1967) establishes a celebrated result stating that a finite observed data set $\{(p^i, x^i) \mid i \in M\}$ is consistent with utility maximization if and only if the observations satisfy GARP. To prove that the observations derived from utility maximization satisfy GARP, the standard approach is to use the non-satiation property of the utility function; see, e.g., Diewert (1973, pp. 420-421), Fostel, Scarf and Todd (2004, p.212), and Varian

[5] Samuelson (1948) introduced a more restrictive axiom than SARP, now known as the Weak Axiom of Revealed Preference.

[6] GARP is equivalent to Afriat (1967)'s Cyclical Consistency.

(1982, p.946). On this point, see Vermeulen (2011, p.4) for a historical account.

As stated earlier, our purpose is to consider the environment where all commodities are inherently indivisible, such as houses and cars. Needless to say, it is more realistic to assume that all goods are traded in integer (or rational) quantities. Thus in the current situation, the consumer's consumption set will be $\mathbb{Z}^n_+$ instead of $\mathbb{R}^n_+$, and his utility function will be $u : \mathbb{Z}^n_+ \to \mathbb{R}$. To make the model even more practical, the price space is also assumed to be $\mathbb{Z}^n_+$ instead of $\mathbb{R}^n_+$. For instance, no unit of a price is less than a penny or cent. Under the current framework, non-satiation is meaningless. This implies that the existing approach of using non-satiation to show that the observations derived from utility maximization satisfy GARP can no longer be applied. To deal with the current model, we first need to modify the standard notion of the consumer's demand set. Given $p \in \mathbb{Z}^n_+$ and budget $b \in \mathbb{Z}_+$, the demand set of the consumer is given by $D_u(p, b) = \arg\max\{u(x) \mid p \cdot x \le b,\ \ x \in \mathbb{Z}^n_+\}$. We refine the demand set $D_u(p, b)$ as follows:

$$D^*_u(p, b) = \{x \in D_u(p, b) \mid p \cdot x \le p \cdot y,\ \forall y \in D_u(p, b)\}$$

That is, $D^*_u(p, b)$ contains those bundles which not only give the consumer the highest utility under his budget but also have the least cost. Any bundle in $D^*_u(p, b)$ will be called an *optimal bundle with tight budget* and $D^*_u(p, b)$ the *tight budget demand set*. In this case, we say that the consumer is a *tight budget utility maximizer*. A tiny step forward as it might appear to be, this refinement is meaningful and natural, more importantly crucial to our analysis on the current discrete model. Of course, this concept can be applied to the continuous case as well for which the non-satiation assumption can be dropped.

The next little example demonstrates that observations derived just from utility maximization without tight budget could violate GARP. Suppose that the consumer faces two indivisible goods and has the utility function of $u(x_1, x_2) = \min\{x_1, x_2\}$ for every $(x_1, x_2) \in \mathbb{Z}^2_+$ and a budget of 32. The prevailing market prices are $p^1 = (10, 11)$ and $p^2 = (11, 10)$, respectively. Then we have possible outcomes $x^1 = (1, 2) \in D_u(p^1, b) = \{(2, 1), (1, 2), (1, 1)\}$ and $x^2 = (2, 1) \in D_u(p^2, b) = D_u(p^1, b)$. Because $p^1 \cdot (x^2 - x^1) = -1 < 0$ and $p^2 \cdot (x^1 - x^2) = -1 < 0$, GARP is violated! However, using the tight budget demand set we have $D^*_u(p^1, b) = \{(1, 1)\} = D^*_u(p^2, b)$, so that outcomes should be $x^1 = x^2 = (1, 1)$. Because $p^1 \cdot (x^2 - x^1) = p^2 \cdot (x^1 - x^2) = 0$, GARP is satisfied! Let us make a comparison with the case of divisible goods. We have the same form of utility function $u(x_1, x_2) = \min\{x_1, x_2\}$ for every $(x_1, x_2) \in \mathbb{R}^2_+$ and the same budget of 32. The same market prices are $p^1 = (10, 11)$ and $p^2 = (11, 10)$, respectively. Note that because goods are perfectly divisible, the consumption space is $\mathbb{R}^2_+$ instead of $\mathbb{Z}^2_+$. In this case we have $D_u(p^1, b) = D^*_u(p^1, b) = D_u(p^2, b) = D^*_u(p^2, b) = \{(\frac{32}{21}, \frac{32}{21})\}$ and GARP is trivially satisfied. Moreover the consumer achieves a higher utility of $\frac{32}{21}$ than 1 in the case of indivisible goods.

The following result shows a benefit of the introduction of the tight budget demand set. Observe that we do not impose any condition on the consumer's utility function $u : \mathbb{Z}^n_+ \to \mathbb{R}$. The proof is quite simple but does make use of the definition of the tight budget demand set.

Lemma 1 *If a finite observed data set $\{(p^i, x^i) \mid i \in M\}$ with $(p^i, x^i) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+$ for all $i \in M$ is derived from tight budget utility maximization, the data set must satisfy GARP.*

Proof. By assumption we know $x^j \in D^*_u(p^j, b_j)$ for all $j = 1, 2, \cdots, m$. Suppose that if $p^j \cdot x^{j+1} \leq p^j \cdot x^j$, then x^{j+1} could have been purchased at prices p^j. Since x^{j+1} was not purchased at p^j, it cannot be strictly preferred to x^j so that $u(x^j) \geq u(x^{j+1})$. The entire sequence of inequalities $u(x^j) \geq u(x^{j+1})$, $j = 1, 2, \cdots, m-1$ implies $u(x^1) \geq u(x^m)$. Suppose to the contrary that $p^m \cdot x^1 < p^m \cdot x^m$. Then $u(x^m) \leq u(x^1)$ together with $p^m \cdot x^1 < p^m \cdot x^m$ would imply $x^m \notin D^*_u(p^m, b_m)$, yielding a contradiction! So $p^m \cdot x^1 \geq p^m \cdot x^m$ and GARP is satisfied. $\square$

A utility function $u : \mathbb{Z}^n_+ \to \mathbb{R}$ is *discrete concave* if, for every $x^1, x^2, \cdots, x^t \in \mathbb{Z}^n_+$ with $t \leq n+1$ and every rational $\lambda_1 \geq 0, \lambda_2, \cdots, \lambda_t \geq 0$ with $\sum_{j=1}^t \lambda_j = 1$ and $\sum_{j=1}^t \lambda_j x^j \in \mathbb{Z}^n_+$, we have $u(\sum_{j=1}^t \lambda_j x^j) \geq \sum_{j=1}^t \lambda_j u(x^j)$.

The following theorem is a discrete analogue of the Afriat's theorem and gives a simple testable necessary and sufficient condition that a finite observed data set must satisfy in order to be consistent with tight budget utility maximization.

Theorem 1 *The observations $(p^j, x^j) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+$ for all $j \in M$ satisfy GARP if and only if there exists a discrete concave and integer-valued utility function that rationalizes the observations in the sense of tight budget utility maximization.*

'If part' is proved in Lemma 1 above. The proof of 'only if' proceeds in several steps. First we construct the data matrix $B = (b(i, j))$ of order m from the observations (p^j, x^j) for all $j \in M$ by defining $b(i, j) = p^i \cdot (x^j - x^i), \forall\, i, j \in M$. Observe that $b(i, i) = 0$ and all $b(i, j)$'s are integral, because x^j's and p^j's are integral.

Following Afriat (1967), let us first assume (in fact later we will show) that there exist integers ψ_1, ψ_2, $\cdots$, ψ_m, and $\beta_1 > 0$, $\beta_2 > 0$, $\cdots$, $\beta_m > 0$ to the following system of linear inequalities—called Afriat inequalities

$$\psi_j \leq \psi_i + \beta_i b(i, j), \;\; \forall\, i, j \in M. \tag{1}$$

Now we define the utility function on $\mathbb{R}^n_+$ by

$$\tilde{u}(x) = \min\{\psi_1 + \beta_1 p^1 \cdot (x - x^1), \cdots, \psi_m + \beta_m p^m \cdot (x - x^m)\}$$

Every term in this expression is linear and hence concave. Thus, $\tilde{u}$, as their point-wise minimum, is also concave. Since all ψ_j, β_j, p^j, and x^j are integral, $\tilde{u}(x)$ is an integer value as long as x is integral. Because $\tilde{u}$ is concave on $\mathbb{R}^n_+$, obviously its restriction on $\mathbb{Z}^n_+$ must be discrete concave and integer-valued. The next two steps show that $\tilde{u}$ rationalizes the observations.

(i). $\tilde{u}(x^j) = \psi_j$ for all $j \in M$. By definition $\tilde{u}(x^j) = \min_{i\in M}\{\psi_i + \beta_i b(i,j)\} = \psi_j$, where the minimum is taken from the Afriat inequalities.

(ii). $p^j \cdot x \leq p^j \cdot x^j$ implies $\tilde{u}(x) \leq \tilde{u}(x^j)$. Note that $\tilde{u}(x) \leq \psi_j + \beta_j p^j \cdot (x - x^j) \leq \psi_j = \tilde{u}(x^j)$, where the first inequality follows from the definition of $\tilde{u}$, the second from the fact that $p^j \cdot x \leq p^j \cdot x^j$ and $\beta_j > 0$, and the last equality from (i).

We have shown that the Afriat inequalities imply the existence of a desirable utility function $\tilde{u}$ rationalizing the observations. We will soon prove that if the observations (p^j, x^j), $j \in M$, satisfy GARP, the Afriat inequalities (1) must have integral solutions ψ^*_1, $\cdots$, ψ^*_m, and $\beta^*_1 > 0$, $\cdots$, $\beta^*_m > 0$.

We use the data matrix $B = (b(i,j))$ to construct a directed graph $G(\beta) = (M, A, \beta)$ with $\beta \in \mathbb{Z}^m_+$, where $M = \{1, 2, \cdots, m\}$ is the set of vertices corresponding to the indices $1, \cdots, m$ of the observations, and for $i, j \in M$ with $i \neq j$ the ordered pair $(i,j) \in A$ is an arc with an integer length or weight $\beta_i b(i,j)$. Here i is the tail and j the head of the arc (i,j). Let $\mathbf{1} = (1, \cdots, 1) \in \mathbb{Z}^m_+$ be the m-vector of all $1's$. In the sequel, we first pay attention to the particular graph $G(\mathbf{1})$.

We need to borrow several basic definitions from graph theory. A *path* in a graph G is an alternating sequence $(i_1, (i_1, i_2), i_2, (i_2, i_3), \cdots, (i_{k-1}, i_k), i_k)$, where i_j, $j = 1, \cdots, k$ are vertices, and (i_j, i_{j+1}), $j = 1, \cdots, k-1$, are arcs in the graph. In this case we also say that there is a path from vertex i_1 to vertex i_k. i_1 is called the starting vertex and i_k the terminal vertex of the path. A path is a *shortest path* from vertex i to vertex j in a graph if the sum of the lengths of all arcs on the path is smallest among all possible paths from i to j in the graph. A path with at least one arc is called a *cycle* if the starting vertex of the path coincides with its terminal vertex and the other vertices are distinct. A cycle is called a *negative (zero, or positive) length cycle* if the sum of the lengths of all arcs in the cycle is strictly less than zero (equal to zero, or strictly greater than zero). We may use C to denote a cycle. For ease of notation, C means simply the collection of all arcs in the cycle C. A (sub)graph H is said to be *strongly connected* if for arbitrary two vertices u, v in the graph H there exists a path in H from u to v. A maximal strongly connected subgraph of a graph G is called a *strongly connected component* of the graph G.

With respect to the graph $G(\mathbf{1})$, we can rephrase the Generalized Axiom of Revealed Preference (GARP) in three slightly different ways. The first was used in Afriat (1967) as *Cyclical Consistency*, the second was given in Piaw and Vohra (2003), and the third is new but similar to the second, and convenient to use in the following proof.

Definition 1 *The data matrix B satisfies GARP if every cycle C in the graph $G(\mathbf{1})$ with $b(i,j) \leq 0$ for all arcs $(i,j) \in C$, implies $b(i,j) = 0$ for all $(i,j) \in C$.*

Definition 2 *The data matrix B satisfies GARP if every negative length cycle in the graph $G(\mathbf{1})$ contains at least one arc of positive length.*

The following definition differs from the second in that it does not need to use the sum of the lengths of all arcs in each cycle but instead it requires that if any cycle contains an arc of negative weight, it should also contain an arc of positive weight.

Definition 3 *The data matrix B satisfies GARP if in the graph $G(\mathbf{1})$ every cycle that contains an arc of negative length must also contain an arc of positive length.*

We are now ready to present a constructive and combinatorial proof which gives explicitly integral solutions $\psi_1^*, \cdots, \psi_m^*$, and $\beta_1^* > 0, \cdots, \beta_m^* > 0$ to the system (1) of Afriat inequalities. As pointed out previously, the basic idea of our proof has been used explicitly in Piaw and Vohra (2003), and also implicitly in Afriat (1967), Diewert (1973), Varian (1982), Fostel, Scarf and Todd (2004). Piaw and Vohra (2003) explicitly used the network structure underlying the Afriat inequalities, whereas Afriat (1967), Diewert (1973), Varian (1982), and Fostel, Scarf and Todd (2004) only explored it implicitly or in a less straightforward way. Here we make the argument very elementary, transparent and accessible without assuming the reader's familiarity with any fundamental result from graph theory or linear programming. Another advantage of the current proof is that it can help the reader have a better understanding of why the original case considered by Afriat (1967) is essential, albeit restrictive in the sense that all $b(i,j)$'s are required to be non-zero.

The proof is based on an algorithm which uses the data matrix B as input and yields integral solutions $\psi_1^*, \cdots, \psi_m^*$, and $\beta_1^* > 0, \cdots, \beta_m^* > 0$ as output. The algorithm goes as follows. (If $b(i,j) \geq 0$ for all $i, j \in M$, then $\psi_i^* = \beta_i^* = 1$ $(\forall i \in M)$ give a feasible solution of Afriat's inequalities, so that we assume $b(i,j) < 0$ for some $i, j \in M$ in the sequel.)

Initialization Use the data matrix B to construct the graph $G(\mathbf{1}) = (M, A, \mathbf{1})$.
Step 1 Remove all arcs (i,j) with positive weight $b(i,j) > 0$ from the graph $G(\mathbf{1})$, resulting in a directed graph $G^-(\mathbf{1})$.
Step 2 Decompose the graph $G^-(\mathbf{1})$ into strongly connected components H_1, H_2, $\cdots$, H_κ, where $H_i's$ are indexed in such a way that if there exists a path from H_i to H_j with $i \neq j$, then $i < j$. If some component H_i contains an arc of negative length, then the observed data is not consistent with GARP, and the algorithm terminates.

Step 3 Choose a sufficiently large integer $L > 0$, e.g., take $L = (m-1)\max_{i,j\in M}\{|b(i,j)| \mid b(i,j) < 0\}$. For every $l = 1, 2, \cdots, \kappa$, let the multiplier β_i^* of every vertex i in the subgraph H_l be equal to $\beta_i^* = L^{l-1}$, the $(l-1)$-th power of L.
Step 4 Use the integers β_i^*, $i \in M$, to construct the graph $G(\beta^*)$. Take $\psi_1^* = 0$. For any $i \in M$ with $i > 1$, let ψ_i^* be equal to the length of a shortest path from vertex 1 to vertex i in the graph $G(\beta^*)$.

The numbering of the strongly connected components H_1, H_2, $\cdots$, H_κ is called a *topological ordering*, and each H_i is an *equivalence class* with respect to the binary relation induced by reachability by paths. Let us illustrate the working of the algorithm by an example. Suppose that the data set is given

$$\{(p^i, x^i) \mid i \in M\} = \{((10,1),(1,2)),((10,11),(1,1)),((1,10),(2,1)),((11,10),(1,1))\},$$

where $M = \{1,2,3,4\}$. Then its corresponding data matrix is

$$B = \begin{bmatrix} 0 & -1 & 9 & -1 \\ 11 & 0 & 10 & 0 \\ 9 & -1 & 0 & -1 \\ 10 & 0 & 11 & 0 \end{bmatrix}.$$

It is easy to check that the graph $G^-(\mathbf{1})$ consists of three strongly connected components H_1 containing vertex 1, H_2 vertex 3, and H_3 vertices 2 and 4. Then we have $\kappa = 3$, $L = 3$, $\beta_1^* = 1$, $\beta_3^* = 3$, and $\beta_2^* = \beta_4^* = 9$. Computing shortest paths from vertex 1 to $i \in M$ in the graph $G(\beta^*)$ yields $\psi_1^* = 0$, $\psi_3^* = 9$, and $\psi_2^* = \psi_4^* = -1$. We could also have another topological ordering due to the fact that in the graph $G^-(\mathbf{1})$, vertices 1 and 3 are not connected. So the graph $G^-(\mathbf{1})$ also consists of three strongly connected components H_1 containing vertex 3, H_2 vertex 1, and H_3 vertices 2 and 4. We have $\kappa = 3$, $L = 3$, $\beta_3^* = 1$, $\beta_1^* = 3$, and $\beta_2^* = \beta_4^* = 9$. Computing shortest paths in the graph $G(\beta^*)$ yields $\psi_1^* = 0$, $\psi_3^* = 27$, and $\psi_2^* = \psi_4^* = -3$.

We are now ready to establish the following general result.

Lemma 2 *Under GARP, the integers $\beta_i^* > 0$ and ψ_i^*, $i \in M$, generated by the algorithm, are the solution to the system of Afriat inequalities.*

Proof. It is easy to see that the graph $G^-(\mathbf{1})$ generated in Step 1 of the algorithm contains no negative length cycle because of GARP, but may contain zero length cycle with all arcs of zero length. Each zero length cycle with all arcs of zero length must be in one of strongly connected components $H_i's$. Notice that due to the decomposition into strongly connected components of $G^-(\mathbf{1})$ there exists no path from H_j to H_i with $j > i$. See, e.g., Fujishige (2005, p.13) on the decomposition of more general graphs.

Next we show that the graph $G(\beta^*)$ contains no negative length cycle. Put $K = \max_{i,j\in M}\{|b(i,j)| \mid b(i,j) < 0\}$ and $L = (m-1)K$. Let C be any cycle in $G(\beta^*)$. If all the vertices of cycle C belong to the vertex set of a single strongly connected component, the length of C is nonnegative. Hence we assume that C contains vertices of at least two strongly connected components. Let i^* be the maximum index i such that H_i contains a vertex of cycle C. Then there exists an arc (y^*, z^*) in C such that y^* belongs to H_{i^*} and z^* to H_{j^*} with $j^* < i^*$. Now suppose that the arcs in C of negative length are given by $(y_1, z_1), \cdots, (y_\ell, z_\ell)$. Note that for each $s = 1, \cdots, \ell$, vertex y_s belongs to H_j with $j < i^*$. Hence,

$$\begin{aligned}\text{the length of } C &\geq \beta^*_{y^*} b(y^*, z^*) + \beta^*_{y_1} b(y_1, z_1) + \cdots + \beta^*_{y_\ell} b(y_\ell, z_\ell) \\ &\geq L^{i^*-1} - \ell K L^{i^*-2} \geq L^{i^*-1} - (m-1)KL^{i^*-2} = 0,\end{aligned}$$

where note that $b(y^*, z^*)$ is a positive integer.

Because the graph $G(\beta^*)$ contains no negative length cycle, for every $i \in M$ with $i > 1$ there exists a shortest path, of length ψ^*_i, from vertex 1 to vertex i and thus ψ^*_i is well-defined and is an integer. Hence we have

$$\psi^*_j \leq \psi^*_i + \beta^*_i b(i,j), \ \ \forall i, j \in M.$$

Observe that the left-hand side is the length of a shortest path from vertex 1 to vertex j and the right-hand side is the length of a path from vertex 1 to vertex j composed of a shortest path from vertex 1 to vertex i and the arc (i, j) from vertex i to vertex j. The definition of a shortest path validates clearly the above inequality for all $i, j \in M$. $\square$

3 Concluding Remarks

We wrap up our discussion with several remarks. Afriat (1967) established his theorem using the method of induction for the special but essential case of all $b(i,j) \neq 0$ with $i \neq j$. This can be seen from our proof, namely, his case will generate exactly m strongly connected components $H_1, H_2, \cdots, H_m$, each consisting of a single vertex, where m is the number of observations.

Diewert (1973) and Varian (1982) studied the general case in which $b(i,j)$'s with $i \neq j$ are allowed to be zero. This case involves the subtle issue of indifference classes in the revealed preference ordering. They considered the binary relation (i, j) meaning $b(i,j) \leq 0$, and examined the transitive closure of the relation and indifference classes. Their indifference classes can be seen as the strongly connected components in our graph $G^-(\mathbf{1})$. While Diewert (1973) found the solution to the system of Afriat inequalities by solving a linear programming problem, in part of his proof Varian (1982) employed a graph-theoretic algorithm for computing the transitive closure of the binary relation. Their proofs also contained an inductive argument and were complex and lengthy.

Fostel, Scarf and Todd (2004) provided two proofs of Afriat's theorem. The first is an induction method and also implicitly uses a structure similar to our graph $G^{-}(\mathbf{1})$. Their second proof makes use of the duality theorem from linear programming. Piaw and Vohra (2003) explored explicitly the network structure inherent in the Afriat inequalities and presented a graph-theoretic constructive proof.

In the current paper we identify a common property—equivalence classes—used explicitly or implicitly in the five previous papers, and make full use of it. In particular, we simplify their approaches by decomposing $G^{-}(\mathbf{1})$ into strongly connected components and taking a topological ordering of the components as $H_1,\ H_2,,\ \cdots, H_\kappa$, from which we can check whether observed data are consistent with GARP, and if consistent, we can compute feasible β_i^* for $i = 1, \cdots, m$. This requires $\mathrm{O}(m^2)$ time, while computing ψ_i^* for $i = 1, \cdots, m$ requires $\mathrm{O}(m^3)$ time shortest path computation.

In summary, our proof is similar to Piaw and Vohra (2003) and also closely related to Afriat (1967), Diewert (1973), Varian (1982), and Fostel, Scarf and Todd (2004). Here we have made the argument more transparent and more accessible without assuming the reader's familiarity with any fundamental result from graph theory or linear programming. In our argument, the explicit use of the decomposition into strongly connected components plays an important role in helping reveal more detailed and more subtle structures of the graph $G(\mathbf{1})$ and simplify the proof considerably. Of course, the very elementary, intuitive and simple proof of Afriat's theorem is merely a byproduct of the current paper whose purpose has been to extend the theory to the equally important case of indivisible goods. We hope that this paper will be of interest and use to researchers who wish to grasp the essence and wide applicability of Afriat's celebrated theorem.

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